Faculty of Exact Sciences
Department of Mathematics

Intersection models

Zhixing You with Assaf Rinot and Jiachen Yuan
zhixingy121@gmail.com

אוניברסיטת בר-אילן
Bar-Ilan University

Intersection models

The method of taking the limit of models of a particular theory has been profoundly used in the study of mathematical logic. Taking the unions of an increasing sequence of generic extensions is central and prevalent in set theory (e.g., iterated forcing). However,the topic of taking intersections was left relatively understudied. The results of this poster demonstrates its utility through applications that are related to weakening of some classical compactness principles.

Sum ultrafilters and an extension of Menas' theorem

Let U be an ultrafilter over an uncountable cardinal κ. For any cardinals $\delta < \lambda < \kappa$, say U is (δ, λ)-*indecomposable* if for any μ with $\delta < \mu < \lambda$ and any $f : \kappa \to \mu$, there exists an $A \in [\mu]^{<\mu}$ such that $f^{-1}[A] \in U$. Say U is *indecomposable* if it is (ω, κ)-indecomposable.
Suppose δ is an infinite regular cardinal, and $\langle \kappa_i \mid i < \delta \rangle$ is a non-decreasing sequence of infinite cardinals. Suppose W is a uniform ultrafilter over δ and U_i is a uniform ultrafilters over κ_i for each $i < \delta$. Then we can define the sum ultrafilter $W\text{-}\lim_{i<\delta} U_i$ over $\sup_{i<\delta} \kappa_i$ by letting $X \in W\text{-}\lim_{i<\delta} U_i$ iff $\{i < \delta \mid X \cap \kappa_i \in U_i\} \in W$.
The approach of intersection of a sequence of outer models helps us to understand interesting ultrafilters.

Theorem. *Suppose δ is a regular cardinal and there exists a decreasing sequence of inner models $\vec{M} := \langle M_i \mid i < \delta \rangle$ with $M := \bigcap_{i<\delta} M_i$ an inner model, two non-decreasing sequences $\vec{\lambda} := \langle \lambda_i \mid i < \delta \rangle$ and $\vec{\kappa} := \langle \kappa_i \mid i < \delta \rangle$ and a sequence $\vec{U} := \langle U_i \mid i < \delta \rangle$ such that*

1. *$W \in M$ is a uniform ultrafilter on δ and $\mathcal{P}(\delta)^{M_0} = \mathcal{P}(\delta)^M$;*
2. *$\vec{M}_{\geq i}$, $\vec{\lambda}_{\geq i}$, $\vec{\kappa}_{\geq i}$ and $\vec{U}_{\geq i}$ are definable in M_i for every $i < \delta$;*
3. *$U_i \in M_i$ is a uniform λ_i-complete ultrafilter over κ_i for every $i < \delta$.*

Then the sum ultrafilter $W\text{-}\lim_{i<\delta} U_i$ is in M and uniform. Moreover,

1. *for any regular $\mu \leq \sup_{i<\delta} \lambda_i$, if W is μ-complete, then $W\text{-}\lim_{i<\delta} U_i$ is μ-complete;*
2. *if $M_0 \models {}^{|\delta|}M \subseteq M$, then $W\text{-}\lim_{i<\delta} U_i$ is $(\delta, \sup_{i<\delta} \lambda_i)$-indecomposable.*

Ketonen's question

Silver asked whether an inaccessible κ carrying a uniform indecomposable ultrafilter is necessarily measurable, and Ketonen gave a partial answer in the affirmative direction, showing that if κ is moreover weakly compact, then it must be a Ramsey cardinal. Motivated by this finding, he asked:

Question (Ketonen, 1980). *Suppose that κ is a weakly compact cardinal carrying a uniform indecomposable ultrafilter. Must κ be measurable?*

A negative answer to Silver's question was soon given by Sheard, but Ketonen's question remained open. In this work we answer it in the negative.

Theorem. *Assuming the consistency of a measurable cardinal, it is consistent that a weakly compact cardinal, say κ, carries an indecomposable ultrafilter, yet it is not measurable.*

A further analysis shows that for any given infinite regular $\delta \leq \kappa$, consistently the following holds:

1. There is no uniform saturated ideal over κ;
2. there is a uniform normal precipitous ideal $\mathcal{I}$ over κ such that $\mathcal{P}(\kappa)/\mathcal{I}$ has a κ-distributive $<\delta$-closed subset, and there is no uniform precipitous ideal $\mathcal{I}$ over κ such that $\mathcal{P}(\kappa)/\mathcal{I}$ is δ-strategically closed.

This gives a new proof of a result of Foreman, Magidor and Zeman that relates to Welch game.

δ-strong compactness

Suppose $\kappa \geq \delta$ are two uncountable cardinals. Say κ is δ-strongly compact iff for every regular $\lambda \geq \kappa$, there is a δ-complete uniform ultrafilter over λ.
This notion is introduced by Bagaria and Magidor and successfully captures a gallery of natural compactness properties occurring in various areas of mathematics. In their study, they proved that the least δ-strongly compact cardinal may be a singular cardinal. Then Gitik proved that the least δ-strongly compact cardinal need not be a large cardinal in the classical sense while still being regular. Later on, You and Yuan obtained a full characterization of the possible cofinalities of the least δ-strongly compact cardinal. In a follow-up paper, they gave a complete picture dealing with *regular* δ-strongly compact cardinals for multiple δ's simultaneously. In contrast, whether there could exist singular δ-strongly compact cardinals for different δ's simultaneously remained open.

Question (Bagaria, Magidor). *Under suitable large cardinal assumption, can there consistently exist two regular cardinals $\delta_0 < \delta_1$ and two singular cardinals $\kappa_0 < \kappa_1$ such that κ_i is the least δ_i-strongly compact cardinal for all $i < 2$?*

We answer this question in the affirmative.

Theorem. *From a suitable large cardinal hypothesis, it is consistent that for proper class many cardinals δ, the least δ-strongly compact cardinal is singular.*

Applications

1. Suppose κ is a supercompact cardinal and $\delta < \kappa$ is a regular cardinal. Via the Bukovský-Dehornoy phenomenon, we can get a suitable iterated ultrapower $\langle M_i, \pi_{i,j} \mid i \leq j \leq \delta \rangle$ such that $M := \bigcap_{i<\delta} M_i[P \restriction i] = M_\delta[P]$ for some δ-sequence P. For every successor $i < \delta$, $\pi_{0,i}(\kappa)$ is supercompact in $M_i[P \restriction i]$. Thus $\pi_{0,\delta}(\kappa)$ is μ-strongly compact in M if δ carries a μ-complete uniform ultrafilter.
2. Suppose for every $i < \delta$, $\mathbb{P}_i$ is a poset. For all $i \leq j < \delta$, the map $\pi_{i,j} : \mathbb{P}_i \to \mathbb{P}_j$ is a projection, and for all $i \leq j \leq k$, $\pi_{i,k} = \pi_{j,k} \circ \pi_{i,j}$. Let G_0 be $\mathbb{P}_0$-generic over V. Let G_i be the filter generated by $\pi_i[G_0]$, then G_i is $\mathbb{P}_i$-generic.

   If $\mathbb{P}_0$ is δ-distributive and $\langle \kappa_i \mid i < \delta \rangle$ is a non-decreasing sequence of cardinals converging to some cardinal κ, with $\kappa_0 > \delta$, then the following holds in $M := \bigcap_{i<\delta} V[G_i]$ (which is a model of ZFC):

   (a) If κ_i is measurable in $V[G_i]$ for every $i < \delta$, then κ carries a (δ, κ)-indecomposable ultrafilter;
   (b) If δ carries a λ-complete uniform ultrafilter in M, and κ_i carries a λ-complete uniform ultrafilter in $V[G_i]$ for each $i < \delta$, then κ carries a λ-complete uniform ultrafilter.

C-sequence number

Our third example is a weakening of weak compactness and concerns the C-*sequence number* due to Lambie-Hanson and Rinot. For a regular uncountable cardinal κ, this cardinal characteristic $\chi(\kappa)$ takes a value in the interval $[0, \kappa]$, and as a rule of thumb, the smaller it is, the more compactness properties κ possesses. By a result of Lambie-Hanson and Rinot, it is consistent for a strongly inaccessible cardinal κ to satisfy $\chi(\kappa) = \delta$ for any prescribed regular $\delta < \kappa$. In contrast, all known consistent examples of $\chi(\kappa)$ being a singular cardinal satisfy that κ is the successor of $\chi(\kappa)$. This raises the following question:

Question (Lambie-Hanson and Rinot, 2019). *Suppose that $\chi(\kappa)$ is a singular cardinal. Must κ be the successor of a cardinal of cofinality $\mathrm{cf}(\chi(\kappa))$?*

We provide a consistent counterexample via intersection models.

Theorem. *For every weakly compact cardinal κ, for every infinite cardinal $\delta \leq \kappa$, there exists a cofinality-preserving forcing extension satisfying $\chi(\kappa) = \delta$. In particular, it is consistent for the C-sequence number of an inaccessible cardinal to be singular.*

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